TRANSITION RADIATION IN ONE-DIMENSIONAL ELASTIC SYSTEMS

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Interest in the interaction of elastic structures with moving objects has increased dramatically with the development of high-speed ground transportation and the increase in the speed of operation of machines and mechanisms (see, for example, [1, 2]). In spite of the many studies which have been performed, however, wave processes which arise in carrying structures, in particular, effects due to wave formation and the accompanying pressure effect of the waves on moving objects, have not been studied [3]. From the practical viewpoint, the case when the structure is substantially nonuniform is of greatest interest. The uniform motion of an object along a structure is accompanied by excitation of elastic waves. It is natural to call this phenomenon, by analogy to the phenomena first described in [4] in application to electromagnetic waves, transition radiation.

In the present paper we give a general formulation of the problem of the interaction of a moving concentrated object with a nonuniform elastic guide [3]. The uniform motion of a body along an infinite string on an elastic base is investigated in detail. It is shown that in the case of motion of a body near a region of nonuniformity (clamping point) transition radiation arises in the string and in the process the waves propagating along the string exert a pressure on the body. An expression relating the work of this force and the energy of the radiation is derived.

1. Consider a one-dimensional elastic system, whose motion is defined in the region D: $\{\alpha \leqslant t \leqslant \beta,\ a \leqslant x \leqslant b\}$ and is characterized by the Lagrangian

$$L = \int_a^b \lambda(x, t)\, dx,$$

where $\lambda = \lambda(t, x, \mathbf{U}(x, t); U_t, U_x)$ the Lagrangian density; $\mathbf{U}(x, t)$ is the vector of generalized coordinates with the components U_1, U_2 ..., U_n, under the assumption that a finite jump in the parameters of this system (for example, the density per unit length, the elasticity of the base, etc.) occurs at x = d (a < d < b).

Let a concentrated object, characterized by the Lagrangian $L^0 = L^0(t, \mathbf{y}(t), \dot{\mathbf{y}}, l(t), \dot{l})$ and vector of generalized coordinates with the components y_1, y_2, ..., y_m, move along an elastic system according to some law $x = l(t)$:

We assume that the elastic system and the motion of the concentrated object are continuous at $x = l(t)$:

$$U_\alpha(t, l(t) - 0) = U_\alpha(t, l(t) + 0) = y_\alpha(t),\ \alpha = 1, p;\ p \leqslant n, m. \tag{1.1}$$

Then, according to the Hamilton—Ostrogradskii principle,

$$\delta \int_\alpha^\beta (L + L^0)\, dt + \int_\alpha^\beta \left(\mathbf{Q}(t)\, \delta y + R \delta l + \delta \int_a^b \mathbf{q}(x, t)\, \mathbf{U}\, dx \right) dt = 0. \tag{1.2}$$

Here $\mathbf{Q}(t)$ and $\mathbf{q}(x, t)$ are, respectively, vectors of the external generalized concentrated and distributed forces and R(t) is the total force (including the external and dissipative forces), which gives rise to the motion according to the law $x = l(t)$.

Performing the variation in Eq. (1.2), applying the condition (1.1), and equating with the same variations, we obtain the boundary-value problem [3, 5]

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$$\frac{\partial\lambda}{\partial U^k}-\frac{\partial}{\partial t}\frac{\partial\lambda}{\partial U_t^k}-\frac{\partial}{\partial x}\frac{\partial\lambda}{\partial U_x^k}+q^k(x,t)=0,\ k=1,n,$$
$$\frac{\partial L^0}{\partial l}-\frac{d}{dt}\frac{\partial L^0}{\partial \dot l}+R(t)=[T-\dot l p]_{x=l(t)},$$
$$\frac{\partial L^0}{\partial y_\alpha}-\frac{d}{dt}\frac{\partial L^0}{\partial \dot y_\alpha}+Q_\alpha(t)=\left[\frac{\partial\lambda}{\partial U_x^\alpha}-\dot l\,\frac{\partial\lambda}{\partial U_t^\alpha}\right]_{x=l(t)},\tag{1.3}$$
$$\frac{\partial L^0}{\partial y_\gamma}-\frac{d}{dt}\frac{\partial L^0}{\partial \dot y_\gamma}+Q_\gamma(t)=0,\quad \gamma=p+1,m,$$
$$[U^k]_{x=d}=\left[\frac{\partial\lambda}{\partial U_x^k}\right]_{x=d}=0,\ U(t,a)=\varphi(t),\ U(t,b)=\psi(t),$$

where $T=\lambda-\sum_{k=1}^{n}U_x^k\frac{\partial\lambda}{\partial U_x^k}$; $p=\sum_{k=1}^{n}U_x^k\frac{\partial\lambda}{\partial U_t^k}$; and $F=-[T-\dot l_p]_{x=l(t)}$ is the pressure force exerted by the waves on the moving object; the brackets denote the difference between the values of the enclosed quantity to the right and left of the indicated value of x. The functions $\varphi(t)$ and $\psi(t)$ are assumed to be given.

The boundary-value problem (1.3) describes the interaction of a moving concentrated object with a nonuniform elastic system. One can see that the oscillations of the object and the guide are coupled. It is especially important to take this factor into account for mechanical systems because real objects have high inertia.

2. As an example, consider Winkler's model of the motion of a body of mass m, located in a gravitational field, along a semiinfinite string on an elastic base (Fig. 1). The Lagrangian of the string and the Lagrangian of the moving concentrated system in this case have the form [3]

$$\lambda(x,t)=\frac{1}{2}(\rho U_t^2-NU_x^2-kU^2);\ L^0(t)=\frac{m}{2}(\dot l^2+\dot y^2-2gy).\tag{2.1}$$

Here U(x, t) is the amplitude of small transverse oscillations of the string; ρ is the density per unit length; N is the tension; and k is the stiffness of the elastic base.

Substituting Eq. (2.1) in Eq. (1.3) we obtain

$$\begin{gathered}U_{tt}-c^2U_{xx}+h^2U=0,\ x\leqslant 0,\ -\infty<t<+\infty,\\ c^2=N/\rho,\ h^2=k/\rho,\ U(t,l(t)-0)=U(t,l(t)+0)=y(t),\\ R(t)-[T-\dot l(t)p]_{x=l(t)}=m\ddot l(t),\ m\ddot y(t)=[NU_x+\rho\dot l(t)U_t]_{x=l(t)}-P,\\ U(t,0)=0,\ U\to 0\ \text{as}\ x-vt\to-\infty,\end{gathered}\tag{2.2}$$

where $T=(1/2)(\rho U_t^2+NU_x^2-kU^2)$; $p=-\rho U_xU_t$; and $P=mg$.

In what follows we shall consider the stationary motion of the body, when $R=[T-vp]_{x=vt}$ and $\dot l=v=\text{const}<c$. As initial conditions we chose a stationary profile, formed in the coordinate system moving with the uniformly moving body, in the case when the string is infinite and the base is uniform [5]:

$$U(x,t)\to-\frac{P}{2\rho h\beta}\exp\left(-\frac{h}{\beta}|x-vt|\right)\ \text{as}\ t\to-\infty.\tag{2.3}$$

Here and below $\beta=\sqrt{c^2-v^2}$.

3. As one can see from Eq. (2.2), if

$$|\ddot y(t)|\ll g\ \forall t\leqslant 0\tag{3.1}$$

the transverse force exerted by the body on the string can be assumed to be constant.

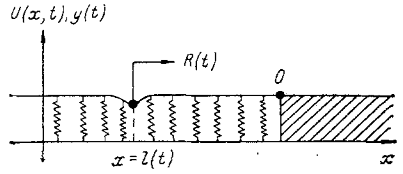

Fig. 1

Assume that the condition (3.1) holds. We now construct the solution of the problem (2.2)-(3.1) using the method of images [6], i.e., we associate to the problem of the motion of two sources along an infinite string lying on a uniform elastic base. We choose the additional source so that the solutions of both problems are identical for $x \leq 0$. It is obvious that the source of the force $-P$, moving according to the law $x = -vt$, satisfies the given condition. Proceeding in this manner and using (2.3), we have the solution of the starting problem (2.2)-(3.1) for $t \leq 0$:

$$U^{-}(x,\, t) = -\frac{P}{2\rho h\beta}\left(\exp\left(-\frac{h}{\beta}\,|x - vt|\right) - \exp\left(\frac{h}{\beta}\,(x + vt)\right)\right),\ x \leqslant 0. \tag{3.2}$$

For $t \geq 0$ we obtain the problem of free oscillations of a string with initial conditions determined by the expression (3.2) as $t \to 0$, i.e., the boundary-value problem

$$\begin{gathered} U_{tt} - c^2 U_{xx} + h^2 U = 0,\ t \geqslant 0,\ x \leqslant 0,\ U(x,\, 0) = 0, \\ U_t(x,\, 0) = (Pv/\rho\beta^2)\exp(hx/\beta),\ U(0,\, t) = 0,\ U \to 0 \text{ as } x \to -\infty. \end{gathered} \tag{3.3}$$

We find the solution of Eq. (3.3) by proceeding similarly, i.e., by constructing an odd continuation of the initial conditions:

$$\begin{gathered} U^{+}(x,\, t) = \frac{P}{\rho h\beta}\left(\exp\left(\frac{h}{\beta}\,x\right)\operatorname{sh}\left(\frac{h}{\beta}\,vt\right) - \operatorname{sh}\left(\frac{h}{\beta}\,(x + vt)\right)\theta(x + ct)\right) + \\ + \frac{Pv}{\pi\rho}\int\limits_{-h}^{h}\frac{\operatorname{sh}\left(\frac{x}{c}\sqrt{h^2 - z^2}\right)\cos(tz)}{z^2\beta^2 + h^2v^2}\,dz\,\theta(x + ct). \end{gathered} \tag{3.4}$$

Here $\theta(x,\, t)$ is the unit step function.

The expressions (3.2) and (3.4) describe transition radiation under the following condition, imposed on the parameters of the initial problem:

$$\alpha^2/(1 - \alpha^2)^{3/2} \ll \rho N/2km,\ \alpha = v/c. \tag{3.5}$$

We now calculate the energy of this radiation in the form $W^r = H(0)$ $\left(H(t) = \frac{1}{2}\int\limits_{-\infty}^{0}(\rho U_t^2 + N U_x^2 + kU^2)\,dx\right.$ is the energy level of the string on the elastic base). As one can see from Eq. (3.2), $U_x(x,\, 0) = 0$, so that, taking into consideration Eq. (3.3), we have

$$W^r = \frac{1}{2}\int\limits_{-\infty}^{0}\rho U_t^2(x,\, 0)\,dx = \frac{P^2v^2}{4h\rho\beta^3}. \tag{3.6}$$

The energy of the radiation increases with the velocity of the body, and hence the probability that the structure collapses also increases. In addition, the energy consumed by the external source which maintains the body in uniform motion in the process of radiation becomes significant.

In this connection it is of interest to find an expression for the pressure force exerted by the waves on the body. The work performed by this force determines the energy consumed by the external source. Taking into consideration the continuity of the string and the boundary conditions at x = vt, we obtain from Eq. (3.2)

$$F = -[T - vp]_{x=vt} = -\frac{P^2}{2\rho\beta^2}\exp\left(\frac{2h}{\beta}vt\right); \tag{3.7}$$

$$W^f = -v\int_{-\infty}^{0} F\,dt = \frac{P^2}{4h\rho\beta}. \tag{3.8}$$

Analysis of the solution found shows that in order to describe completely the energy conversion process in transition radiation the change in the energy of the string deflection moving together with the source ΔH and the work performed by gravity A^g must be taken into account. Indeed, calculating the values of ΔH and A^g in the form

$$\Delta H = 0 - \lim_{t\to-\infty} H(t) = -\frac{P^2c^2}{4h\rho\beta^3}, \tag{3.9}$$
$$A^g = P\left(0 - \lim_{t\to-\infty} U(vt, t)\right) = -\frac{P^2}{2\rho h\beta}$$

and comparing Eqs. (3.6), (3.8), and (3.9) we obtain an expression relating W^f and W^r. This relation is an integral law expressing the change in energy accompanying transition radiation:

$$A^g + W^f - \Delta H = W^r. \tag{3.10}$$

As one can see from Eq. (3.10), when the body moves near the clamp, the energy of the string deflection, moving together with the object, is converted into radiation energy. In the process, work is performed not only by the external source, which mantains the object in uniform motion (as happens in electrodynamic systems [4]), but also by the gravitational field, which acts in a direction transverse to the direction of motion of the body.

4. We now consider the solution of the initial problem, making the assumption that the inertia of the body cannot be neglected compared with the weight of the body (the condition (3.1) is not satisfied). Using once again the method of images, we associate this problem to the auxiliary problem of the motion of two sources of a transverse force: the real source $P_1 = m(g + \ddot{y}(t))$, moving according to the law x = vt, and a fictitious source $P_2 = -P_1$, moving according to the law x = −vt along an infinite string on a uniform elastic base:

$$U_{tt} - c^2U_{xx} + h^2U = 0,\ t \leqslant 0,\ -\infty < x < +\infty; \tag{4.1}$$
$$U(t, vt - 0) = U(t, vt + 0) = y(t),\ [U(x, t)]_{x=-vt} = 0, \tag{4.2}$$
$$[NU_x + \rho vU_t]_{x=\pm vt} = \pm m(g + \ddot{y}),\ R(t) = [T - vp]_{x=vt},$$
$$U \to 0 \ \text{ as } \ x \pm vt \to \pm\infty,$$
$$U \to -\frac{P}{2\rho h\beta}\exp\left(-\frac{h}{\beta}|x - vt|\right) \ \text{ as } \ t \to -\infty.$$

We apply to Eq. (4.1) the Fourier transform

$$V(k, t) = \int_{-\infty}^{+\infty} U(x, t)\exp(ikx)\,dx.$$

Using the operational equations

$$U_{tt} \doteq V_{tt} - \frac{mv^2}{\rho\beta^2}(g + \ddot{y})(\exp(ikvt) - \exp(-ikvt)),$$

$$U_{xx} \doteq -k^2 V - \frac{m}{\rho\beta^2}(g + \ddot{y})(\exp(ikvt) - \exp(-ikvt)),$$

written so as to take into account the conditions (4.2), we obtain for the transform

$$V_{tt} + (c^2k^2 + h^2)V = \frac{m}{\rho}(g + \ddot{y})(\exp(-ikvt) - \exp(ikvt)). \tag{4.3}$$

We now seek the solution of Eq. (4.3) as $V = V^0 + V^1$, where V^0 is the solution of the equation

$$V^0_{tt} + (c^2k^2 + h^2)V = \frac{m}{\rho} g(\exp(-ikvt) - \exp(ikvt)),$$

the inverse transform of which is determined by the expression (3.2).

It is obvious that in this case V^1 will satisfy the equation

$$V^1_{tt} + (c^2k^2 + h^2)V^1 = \frac{m}{\rho}\ddot{y}(\exp(-ikvt) - \exp(ikvt)) \tag{4.4}$$

with the initial conditions

$$V^1 = V^1_t = 0 \quad \text{as} \quad t \to -\infty. \tag{4.5}$$

The general solution of Eq. (4.4) with the initial conditions (4.5) has the form

$$V^1(k, t) = \frac{m}{\rho}\int\limits_{-\infty}^{t} \ddot{y}(\tau)(\exp(-ikv\tau) - \exp(ikv\tau))\frac{\sin\left((t-\tau)\sqrt{h^2 + c^2k^2}\right)}{\sqrt{h^2 + c^2k^2}}\, d\tau.$$

We now switch to the inverse transform. Using the formula [7]

$$\frac{1}{\pi}\int\limits_{0}^{+\infty} \cos(kf)\frac{\sin\left(a\sqrt{k^2 + b^2}\right)}{\sqrt{k^2 + b^2}}\, dk = \frac{1}{2} J_0\left(b\sqrt{a^2 - f^2}\right)\theta(a - |f|)$$

where θ is the unit step function and J_0 is a zeroth order Bessel function, we obtain

$$U^1(x, t) = \frac{m}{2c\rho}\int\limits_{-\infty}^{t} \ddot{y}(\tau)\Big(J_0\left(\frac{h}{c}\sqrt{c^2(t-\tau)^2 - (x + v\tau)^2}\right)\theta\left(c(t-\tau) - |x + v\tau|\right) -$$

$$- J_0\left(\frac{h}{c}\sqrt{c^2(t-\tau)^2 - (x - v\tau)^2}\right)\theta\left(c(t-\tau) - |x - v\tau|\right)\Big)\, d\tau.$$

Finally, using Eq. (3.2), for $t \leqslant 0,\ x \leqslant 0$ we find

$$U(x, t) = U^0(x, t) + U^1(x, t) = -\frac{m}{2\rho h\beta} g\left(\exp\left(-\frac{h}{\beta}|x - vt|\right) - \exp\left(\frac{h}{\beta}(x + vt)\right)\right) +$$

$$+ \frac{m}{2c\rho}\begin{cases} \int\limits_{-\infty}^{\mu^+/\Delta^-} \ddot{y}(\tau) J_0(\nu^+)\, d\tau - \int\limits_{-\infty}^{\mu^+/\Delta^+} \ddot{y}(\tau) J_0(\nu^-)\, d\tau, & x \leqslant vt, \\ \int\limits_{-\infty}^{\mu^+/\Delta^-} \ddot{y}(\tau) J_0(\nu^+)\, d\tau - \int\limits_{-\infty}^{-\mu^-/\Delta^-} \ddot{y}(\tau) J_0(\nu^-)\, d\tau, & x \geqslant vt, \end{cases} \tag{4.6}$$

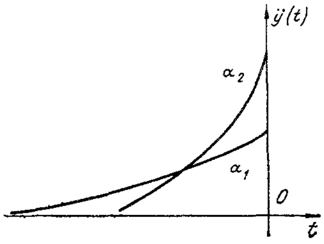

Fig. 2

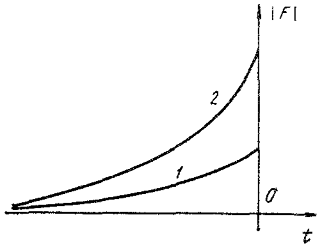

Fig. 3

where $\mu^{\pm} = x \pm ct$; $\Delta^{\pm} = c \pm v$; $\nu^{\pm}(x, t, \tau) = (h/c)\sqrt{c^2(t-\tau)^2 - (x \pm v\tau)^2}$.

In order to determine the unknown $\ddot{y}(t)$ we use the condition that the motion of the body is continuous $y(t) = U(vt, t)$:

$$y(t) = -\frac{m}{2\rho h\beta} g\left(1 - \exp\left(\frac{2h}{\beta} vt\right)\right) + \frac{m}{2c\rho}\left(\int_{-\infty}^{t\Delta^+/\Delta^-} \ddot{y}(\tau) J_0\left(\nu^+(vt, t, \tau)\right) d\tau - \int_{-\infty}^{t} \ddot{y}(\tau) J_0\left(\nu^-(vt, t, \tau)\right) d\tau\right), \quad t \leqslant 0. \qquad (4.7)$$

The integrodifferential equation (4.7) obtained by differentiation reduces to a Volterra integral equation of the second kind for $\ddot{y}(t)$, which can be solved conveniently on a computer.

The function $\ddot{y}(t)$ for different values of $\alpha = v/c < 1$ ($\alpha_1 < \alpha_2$), which was found by numerically integrating Eq. (4.7), has the qualitative form shown in Fig. 2. The solution (4.6) and (4.7) makes it possible to determine the time dependence of the pressure force exerted by the waves on the moving body. Figure 3 shows for fixed parameters of the problem the qualitative time dependence: the curve 1 is the pressure force F exerted by the waves on the body and was found by neglecting the inertia of the body (according to Eq. (3.7)): the curve 2 is the force F obtained from Eqs. (4.6) and (4.7). One can see that taking into account the inertia of the body increases F. Therefore the energy consumed by the external source which maintains the body in uniform motion also increases. The equation of energy balance (3.10), in which only the quantities W^f and W^r change when the inertia of the body is taken into account, allows us to assert that the energy of the transition radiation also increases.

Thus we have shown that when a moving object interacts with nonuniform carrying structures the transition radiation is a characteristic source of vibrations. The analysis performed also shows that the energy consumed on radiation can be equal to the energy of the translational motion of the object and the pressure force of the waves which acts on the moving object in the process of radiation can be impulsive, especially near regions where the change in the parameters of the elastic structure is significant (in particular, near clamps).

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